Gaussian Process Regression on Nested Spaces

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Clermont-Ferrand, Thursday 21 December 2023
Motivation - Cooling system
Motivation - Cooling System

Sections de pale
Motivation—Cooling System

Sections de pale

Calage et longueur de corde
Motivation-Cooling System

Sections de pale

Calage et longueur de corde
Motivation-Cooling System

Sections de pale

Calage et longueur de corde

Empilement tangentiel
Motivation-Cooling System

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Calage et longueur de corde

Empilement tangentiel
Gaussian Process Regression on Nested Spaces

Motivation - Cooling System

Outputs

- Pressure difference: $\Delta P$ (en Pa)
Exploitation of the simulation

Computationally expensive code ⇒ need a metamodel.
Exploitation of the simulation

Computationally expensive code $\Rightarrow$ need a metamodel.

Problem: a lot of inputs (Stagger, chord length, sweep, ...)

Which study?

Until now, at each step sequentially releasing variables by group

$\Rightarrow$ First we consider only stagger.

$\Rightarrow$ built a DoE and a metamodel.

$\Rightarrow$ we add chord length.

$\Rightarrow$ built a new DoE and a new metamodel.

$\Rightarrow$ then the sweeps...
Exploitation of the simulation

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► then the sweeps...
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*Problem*: a lot of inputs (*Stagger, chord length, sweep, ...*)

**Which study?** Until now at each step sequentially releasing variables by group
  - First we consider only staggers ⇒ built a DoE and a metamodel.
  - We add chord length ⇒ built a new DoE and a new metamodel.
  - Then the sweeps...

**Which metamodel adapted to this sequential study?**

Aim: Using all the DoEs.
Example 2D

Let $f$ a function of 2 variables $f(x_1, x_2) = \exp\left(\frac{(x_1-0.2)^2 + (x_2-0.4)^2}{0.3}\right)$.
Example 2D

Let $f$ a function of 2 variables $f(x_1, x_2) = \exp \left( \frac{(x_1 - 0.2)^2 + (x_2 - 0.4)^2}{0.3} \right)$.

What kind of metamodel?
Outline

seqGPR model
  Kriaking
  New Model
  Correction process

Parameters estimation

Applications
  Analytic example in dimension 4
  Industrial Test Cases

Conclusion
Table des matières

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  New Model
  Correction process

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Conclusion
Classical Gaussian process regression [Santner et al., 2003]

The output $f$ is the realization of a Gaussian process

$$Y(x_1, x_2) \sim \mathcal{P}G(m, k)$$
Classical Gaussian process regression [Santner et al., 2003]

The output $f$ is the realization of a Gaussian process

$$Y(x_1, x_2) \sim \mathcal{PG}(m, k)$$

The DoE $(X, y)$ avec $X = X_1 \cup X_2$. 
Classical Gaussian process regression [Santner et al., 2003]

The output $f$ is the realization of a Gaussian process

$$Y(x_1, x_2) \sim \mathcal{PG}(m, k)$$

The DoE $(\mathbf{X}, \mathbf{y})$ avec $\mathbf{X} = \mathbf{X}_1 \cup \mathbf{X}_2$.

Prediction :

$$\hat{y}(x_1, x_2) = m + k((x_1, x_2), \mathbf{X})k(\mathbf{X}, \mathbf{X})^{-1}(\mathbf{y} - m)$$
Parameters inference

\[ Y \sim \mathcal{P}G(m, k(\sigma^2, \theta)) \] with

\[ k(((x_1, x_2), (x'_1, x'_2))) = \sigma^2 \prod_{i=1}^{2} \left( 1 + \frac{\sqrt{5}|x_i - x'_i|}{\theta_i} + \frac{5(x_i - x'_i)^2}{3\theta_i^2} \right) \exp \left( -\frac{\sqrt{5}|x_i - x'_i|}{\theta_i} \right) \]
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\]

\[ \eta = (m, \sigma^2, \theta_1, \theta_2) \] estimated from the DoE \((X, y)\) (size \(n_X\)) by maximum of likelihood
Parameters inference

\[ Y \sim \mathcal{PG}(m, k(\sigma^2, \theta)) \] with

\[
k(((x_1, x_2), (x_1', x_2')) = \sigma^2 \prod_{i=1}^{2} \left( 1 + \frac{\sqrt{5} |x_i - x_i'|}{\theta_i} + \frac{5(x_i - x_i')^2}{3\theta_i^2} \right) \exp \left( -\frac{\sqrt{5} |x_i - x_i'|}{\theta_i} \right)
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\[ \eta = (m, \sigma^2, \theta_1, \theta_2) \] estimated from the DoE \((\mathbf{X}, \mathbf{y})\) (size \(n_\mathbf{X}\)) by maximum of likelood

\[
\mathcal{L}(Y(\mathbf{X}) = \mathbf{y}; \eta) = \frac{1}{(2\pi)^{n_\mathbf{X}/2} |k(\mathbf{X}, \mathbf{X})|^{1/2}} \exp \left( -\frac{(\mathbf{y} - m)^T k(\mathbf{X}, \mathbf{X})^{-1} (\mathbf{y} - m)}{2} \right)
\]
seqGPR model

Can we do better?
seqGPR model

Can we do better? → new Model seq GPR
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seqGPR can be view as a "multi-fidelity model"
[Kennedy and O’Hagan, 2000],
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\[ Y_2(x_1, x_2) = Y_1(x_1) + Z_2(x_1, x_2) \]
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Y_2(x_1, x_2) = Y_1(x_1) + Z_2(x_1, x_2)
\]

\( Y_1 \sim \mathcal{P}G(m, k_1(x_1, x_1')) \) with \( k_1 \) stationary

\( Z_2 \sim \mathcal{P}G(0, k_2((x_1, x_2), (x_1', x_2')) \)
seqGPR model

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- $Z_2 \sim \mathcal{P}G(0, k_2((x_1, x_2), (x'_1, x'_2)))$
- $Z_2(x_1, 0) = 0$
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[Kennedy and O’Hagan, 2000], $f$ realization of:

$$Y_2(x_1, x_2) = Y_1(x_1) + Z_2(x_1, x_2)$$

- $Y_1 \sim PG(m, k_1(x_1, x_1'))$ with $k_1$ stationary
- $Z_2 \sim PG(0, k_2((x_1, x_2), (x_1', x_2')))$
- $Z_2(x_1, 0) = 0$
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$f$ a realisation of $Y_2 \sim PG(m, k_1(x_1, x_1) + k_2((x_1, x_2), (x_1', x_2')))$
seqGPR model

Aim: Build $Z^2$ such that $Z^2(x_1, 0) = \frac{11}{32}$
seqGPR model

\[ Y_1 \]
\[ x_2 \quad x_1 \]
\[ Z_2 \]
\[ x_2 \quad x_1 \]
\[ Y_2 \]
\[ x_2 \quad x_1 \]

**Etape 1:**
\[ f(x_1, 0) \] réalisation de \( Y_1(x_1) \)

**Processus correctif:**
\[ Z_2(x_1, x_2) \]

**Etape 2:**
\[ f(x_1, x_2) \] réalisation de
\[ Y_2(x_1, x_2) = Y_1(x_1) + Z_2(x_1, x_2) \]
seqGPR model

Aim: Build $Z_2$ such that $Z_2(x_1, 0) = 0$?
Idea

Build $Z_2(x_1, x_2)$ such that $Z_2(x_1, 0) = 0$ form a latent process $\tilde{Z}_2 \sim \mathcal{P}G(0, \kappa)$ with a kernel $\kappa$ fixed.
Red (reduced) process)

Red process is defined by $Z_2^{Red}(x_1, x_2) = \tilde{Z}_2(x_1, x_2) - \tilde{Z}_2(x_1, 0)$.
Red (reduced) process

Red process is defined by \( Z_{2}^{\text{Red}}(x_1, x_2) = \tilde{Z}_2(x_1, x_2) - \tilde{Z}_2(x_1, 0) \).

\[
\tilde{Z}_2^{\text{Red}} \sim \mathcal{PG}(0, k_2)
\]
Red (reduced) process)

Red process is defined by $Z_{2}^{Red}(x_1, x_2) = \tilde{Z}_2(x_1, x_2) - \tilde{Z}_2(x_1, 0)$.

$Z_{2}^{Red} \sim \mathcal{PG}(0, k_2)$ with $k_2((x_1, x_2), (x'_1, x'_2)) = \kappa((x_1, x_2), (x'_1, x'_2)) + \kappa((x_1, 0), (x'_1, 0)) - \kappa((x_1, 0), (x'_1, x'_2)) - \kappa((x_1, x_2), (x'_1, 0))$. 
P (preconditioned) process

The process $P$ is defined using conditional expectation introduced by [Gauthier and Bay, 2012]

$$Z^P_2(x_1, x_2) = \tilde{Z}_2(x_1, x_2) - \mathbb{E} \left[ \tilde{Z}_2(x_1, x_2) \mid \tilde{Z}(D) \right]$$

with

$$D = \{(t_1, 0), t_1 \in [0, 1]\}$$
P (preconditioned) process

The process P is defined using conditional expectation introduced by [Gauthier and Bay, 2012]

\[ Z_2^P(x_1, x_2) = \tilde{Z}_2(x_1, x_2) - \mathbb{E} \left[ \tilde{Z}_2(x_1, x_2) \mid \tilde{Z}(\mathcal{D}) \right] \]

with

\[ \mathcal{D} = \{(t_1, 0), t_1 \in [0, 1]\} \]

Proposition

*If the process \( \tilde{Z}_2 \sim \mathcal{P}\mathcal{G}(0, \kappa) \) has a tensor product kernel*

\[ \kappa((x_1, x_2), (x'_1, x'_2)) = \kappa_1(x_1, x'_1)\kappa_2(x_2, x'_2) \]

*Then the conditional expectation is equal to:*

\[ \mathbb{E} \left[ \tilde{Z}_2(x_1, x_2) \mid \tilde{Z}(\mathcal{D}) \right] = \kappa_2(x_2, 0)\tilde{Z}_2(x_1, 0) \]
P (preconditioned) process

If \( \tilde{Z}_2 \sim \mathcal{PG}(0, \kappa) \) with \( \kappa((x_1, x_2), (x'_1, x'_2)) = \kappa_1(x_1, x'_1)\kappa_2(x_2, x'_2) \),
P (preconditioned) process

If $Z_2 \sim \mathcal{P}\mathcal{G}(0, \kappa)$ with $\kappa((x_1, x_2), (x'_1, x'_2)) = \kappa_1(x_1, x'_1)\kappa_2(x_2, x'_2)$, the P process is defined by $Z_2^P(x_1, x_2) = Z_2(x_1, x_2) - \kappa_2(x_2, 0)Z_2(x_1, 0)$. 

Processus Latent : $\tilde{Z}_2(x_1, x_2)$

Projection sur la droite: $\kappa_2(x_2, 0)\tilde{Z}_2(x_1, 0)$

Processus P : $Z_2^P(x_1, x_2)$
P (preconditioned) process

If \( \tilde{Z}_2 \sim \mathcal{P}\mathcal{G}(0, \kappa) \) with \( \kappa((x_1, x_2), (x'_1, x'_2)) = \kappa_1(x_1, x'_1)\kappa_2(x_2, x'_2) \), the P process is defined by \( Z^P_2(x_1, x_2) = \tilde{Z}_2(x_1, x_2) - \kappa_2(x_2, 0)\tilde{Z}_2(x_1, 0) \).

\[
Z^P_2 \sim \mathcal{P}\mathcal{G}(0, \kappa_2)
\]
P (preconditioned) process

If $\tilde{Z}_2 \sim \mathcal{P}G(0, \kappa)$ with $\kappa((x_1, x_2), (x_1', x_2')) = \kappa_1(x_1, x_1')\kappa_2(x_2, x_2')$, the P process is defined by $Z_2^P(x_1, x_2) = \tilde{Z}_2(x_1, x_2) - \kappa_2(x_2, 0)\tilde{Z}_2(x_1, 0)$. 

$Z_2^P \sim \mathcal{P}G(0, k_2)$ with $k_2((x_1, x_2), (x_1', x_2')) = \kappa_1(x_1, x_1')[\kappa_2(x_2, x_2') - \kappa_2(x_2, 0)\kappa_2(x_2', 0)]$. 

Processus Latent: $\tilde{Z}_2(x_1, x_2)$

Projection sur la droite: $\kappa_2(x_2, 0)\tilde{Z}_2(x_1, 0)$

Processus P: $Z_2^P(x_1, x_2)$
Interpretation of the correction process

Latent process $\tilde{Z}_2 \sim \mathcal{GP}(0, \kappa)$
Interpretation of the correction process

Latent process $\tilde{Z}_2 \sim \mathcal{GP}(0, \kappa)$

- Red process:

  \[
  Z_{2}^{\text{Red}}(x_1, x_2) = \tilde{Z}_2(x_1, x_2) - \tilde{Z}_2(x_1, 0)
  \]

  $\implies$ global disruption
Interpretation of the correction process

Latent process $\tilde{Z}_2 \sim \mathcal{GP}(0, \kappa)$

- Red process:

$$Z_{2}^{Red}(x_1, x_2) = \tilde{Z}_2(x_1, x_2) - \tilde{Z}_2(x_1, 0)$$

$\implies$ global disruption

- P process:

$$Z_{2}^{P}(x_1, x_2) = \tilde{Z}_2(x_1, x_2) - \kappa_2(x_2, 0)\tilde{Z}_2(x_1, 0)$$

$\implies$ local disruption
Table des matières

seqGPR model
  Kriking
  New Model
  Correction process

Parameters estimation

Applications
  Analytic example in dimension 4
  Industrial Test Cases

Conclusion
Estimation Problem
Estimation Problem

- Model: \( Y_2(x_1, x_2) = Y_1(x_1) + Z_2(x_1, x_2) \) with
  - \( Y_1 \sim \mathcal{P}G(m, k_1), \) \( k_1 \) stationary
  - \( Z_2 \sim \mathcal{P}G(0, k_2), \) Red or P \( \Rightarrow Z_2(x_1, 0) = 0 \)
  - \( Z_2 \perp Y_1 \)
Estimation Problem

Model: \( Y_2(x_1, x_2) = Y_1(x_1) + Z_2(x_1, x_2) \) with

- \( Y_1 \sim PG(m, k_1) \), \( k_1 \) stationary
- \( Z_2 \sim PG(0, k_2) \), Red or P \( \Rightarrow Z_2(x_1, 0) = 0 \)
- \( Z_2 \perp Y_1 \)

We need to estimate \( \eta = (\eta_1, \eta_2) \) with \( \eta_1 \) parameters of \( Y_1 \) and \( \eta_2 \) parameters of \( Z_2 \). We have \((X_1, y_1)\) and \((X_2, y_2)\).
Estimation Problem

- Model: \( Y_2(x_1, x_2) = Y_1(x_1) + Z_2(x_1, x_2) \) with
  - \( Y_1 \sim \mathcal{PG}(m, k_1) \), \( k_1 \) stationary
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- We need to estimate \( \eta = (\eta_1, \eta_2) \) with \( \eta_1 \) parameters of \( Y_1 \) and \( \eta_2 \) parameters of \( Z_2 \). We have \( (X_1, y_1) \) and \( (X_2, y_2) \).

- Maximum of (Log-)likelihood \( \mathcal{L}(Y_1(X_1) = y_1, Y_2(X_2) = y_2; \eta) \)
Nested Designs

Decoupled Log-likelihood:

1. $\eta_1$: parameters off $Y_1$
   $\max_{\eta_1} \mathcal{L}(Y_1(X_1) = y_1; \eta_1)$

2. $\eta_2$: parameters on $Z_2$
   $\max_{\eta_2} \mathcal{L}(Z_2(X_2) = z_2; \eta_2)$
Non-Nested designs

Problem: \( \mathcal{L}(Y_1(\mathbf{X}_1) = y_1, Y_2(\mathbf{X}_2) = y_2; \eta) \) can’t be decoupled.
Non-Nested designs

Problem: \( \mathcal{L}_1(Y_1(X_1)) = y_1, Y_2(X_2) = y_2; \eta \) can’t be decoupled.

Complete data s.t. \( X_2 \) is nested in \( \tilde{X}_1 = X_1 \cup \hat{X}_1 \).
Non-Nested designs

Problem: \( \mathcal{L}\mathcal{L}(Y_1(X_1)) = y_1, Y_2(X_2) = y_2; \eta \) can’t be decoupled.

Complete data s.t \( X_2 \) is nested in \( \tilde{X}_1 = X_1 \cup \hat{X}_1 \).

\[ \implies \mathcal{L}\mathcal{L}(Y_1(\tilde{X}_1) = \tilde{y}_1, Y_2(X_2) = y_2; \eta) \]
can be now decoupled in \( \mathcal{L}\mathcal{L}(Y_1(\tilde{X}_1) = \tilde{y}_1; \eta_1)\mathcal{L}\mathcal{L}(Z_2(X_2) = z_2; \eta_2) \)
Expectation-Maximization (EM) [Hastie et al., 2009]

EM algorithm is defined by sequences
\[(\hat{L}^{(i)}(Y_1(\tilde{X}_1) = \tilde{y}_1; \eta_1))_i, \ (\hat{L}^{(i)}(Z_2(X_2) = z_2; \eta_2))_i, \text{ et } (\eta^{(i)})_i.\]
Expectation-Maximization (EM) [Hastie et al., 2009]

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\( (\hat{L}^{(i)}(Y_1(\tilde{X}_1) = \tilde{y}_1; \eta_1))_i, (\hat{L}^{(i)}(Z_2(\tilde{X}_2) = z_2; \eta_2))_i, \) et \((\eta^{(i)})_i\).

At each step \( i + 1 \),
Expectation-Maximization (EM) [Hastie et al., 2009]

EM algorithm is defined by sequences

\[(\hat{\mathcal{L}}^{(i)}(Y_1(\tilde{X}_1) = \tilde{y}_1; \eta_1))_i, (\hat{\mathcal{L}}^{(i)}(Z_2(X_2) = z_2; \eta_2))_i, \text{ et } (\eta^{(i)})_i.\]

At each step \(i + 1\),

- **Expectation**:
  \[
  \hat{\mathcal{L}}^{(i+1)}(Y_1(\tilde{X}_1) = \tilde{y}_1; \eta_1) = \mathbb{E}_{\eta^{(i)}}[\mathcal{L}\mathcal{L}(Y_1(\tilde{X}_1) = \tilde{y}_1; \eta_1) \mid Y_1(\tilde{X}_1) = y_1, Y_2(X_2) = y_2]
  \]
  \[
  \hat{\mathcal{L}}^{(i+1)}(Z_2(\tilde{X}_2) = z_2; \eta_2) = \mathbb{E}_{\eta^{(i)}}[\mathcal{L}\mathcal{L}(Z_2(\tilde{X}_2) = z_2; \eta_2) \mid Y_1(\tilde{X}_1) = y_1, Y_2(X_2) = y_2]
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Expectation-Maximization (EM) [Hastie et al., 2009]

EM algorithm is defined by sequences

\((\hat{\mathcal{L}}^{(i)}(Y_1(\tilde{X}_1) = \tilde{y}_1; \eta_1))_i, \ (\hat{\mathcal{L}}^{(i)}(Z_2(\tilde{X}_2) = z_2; \eta_2))_i, \) et \((\eta^{(i)})_i\).

At each step \(i + 1\),

- **Expectation** :
  \[- \hat{\mathcal{L}}^{(i+1)}(Y_1(\tilde{X}_1) = \tilde{y}_1; \eta_1) = \mathbb{E}_{\eta^{(i)}}[\mathcal{L}(Y_1(\tilde{X}_1) = \tilde{y}_1; \eta_1) | Y_1(\tilde{X}_1) = y_1, Y_2(\tilde{X}_2) = y_2] \]
  \[- \hat{\mathcal{L}}^{(i+1)}(Z_2(\tilde{X}_2) = z_2; \eta_2) = \mathbb{E}_{\eta^{(i)}}[\mathcal{L}(Z_2(\tilde{X}_2) = z_2; \eta_2) | Y_1(\tilde{X}_1) = y_1, Y_2(\tilde{X}_2) = y_2] \]

- **Maximization** :
  \[- \eta_1^{(i+1)} : \max_{\eta_1} \hat{\mathcal{L}}^{(i+1)}(Y_1(\tilde{X}_1) = \tilde{y}_1; \eta_1) \]
  \[- \eta_2^{(i+1)} : \max_{\eta_2} \hat{\mathcal{L}}^{(i+1)}(Z_2(\tilde{X}_2) = z_2; \eta_2) \]
Table des matières

- seqGPR model
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Applications
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Conclusion
Function and data

Let the function

$$f(x_1, x_2, x_3, x_4) = f_1(x_1, x_2) + f_2(x_1, x_2, x_3, x_4)$$

with

$$\begin{cases}
    f_1(x_1, x_2) &= \left[ 4 - 2.1(4x_1 - 2)^2 + \frac{(4x_1 - 2)^4}{3} \right] (4x_1 - 2)^2 \\
    &+ (4x_1 - 2)(2x_2 - 1) + [-4 + 4(2x_2 - 1)^2] (2x_2 - 1)^2 \\
    f_2(x_1, x_2, x_3, x_4) &= 4 \exp \left(-\|x - 0.3\|^2\right)
\end{cases}$$
Function and data

Let the function

\[ f(x_1, x_2, x_3, x_4) = f_1(x_1, x_2) + f_2(x_1, x_2, x_3, x_4) \]

with

\[
\begin{align*}
    f_1(x_1, x_2) &= \left[ 4 - 2.1(4x_1 - 2)^2 + \frac{(4x_1 - 2)^4}{3} \right] (4x_1 - 2)^2 \\
    &\quad + (4x_1 - 2)(2x_2 - 1) + \left[ -4 + 4(2x_2 - 1)^2 \right] (2x_2 - 1)^2 \\
    f_2(x_1, x_2, x_3, x_4) &= 4 \exp \left( -\|x - 0.3\|^2 \right) 
\end{align*}
\]

- At step 1, let the restriction \( f(x_1, x_2, \frac{x_1 + x_2}{2}, 0.2x_1 + 0.7) \). Computer code evaluations at DoE (\( X_1, y_1 \)) of size 20.
Function and data

Let the function

\[ f(x_1, x_2, x_3, x_4) = f_1(x_1, x_2) + f_2(x_1, x_2, x_3, x_4) \]

with

\[
\begin{cases}
  f_1(x_1, x_2) = \left[ 4 - 2.1(4x_1 - 2)^2 + \frac{(4x_1 - 2)^4}{3} \right] (4x_1 - 2)^2 \\
  + (4x_1 - 2)(2x_2 - 1) + [-4 + 4(2x_2 - 1)^2] (2x_2 - 1)^2 \\
  f_2(x_1, x_2, x_3, x_4) = 4 \exp \left( -\|x - 0.3\|^2 \right) 
\end{cases}
\]

- At step 1, let the restriction \( f(x_1, x_2, \frac{x_1 + x_2}{2}, 0.2x_1 + 0.7) \). Computer code evaluations at DoE \((X_1, y_1)\) of size 20.
- At step 2, new simulations of \( f \) at points DoE \((X_2, y_2)\), a design in dimension 4 and of size 40.
Benchmark models

- **K_{tot}**: Kriging ($Y \sim \mathcal{PG}(m, k)$) trained on ($X_1, y_1$) and ($X_2, y_2$).
Benchmark models

- **K_tot** : Kriging ($Y \sim \mathcal{PG}(m, k)$) trained on $(X_1, y_1)$ and $(X_2, y_2)$.
- **K_2** : Kriging ($Y \sim \mathcal{PG}(m, k)$) trained on $(X_2, y_2)$.
Benchmark models

- **K\_tot**: Kriging \( (Y \sim \mathcal{P}G(m, k)) \) trained on \((X_1, y_1)\) and \((X_2, y_2)\).
- **K\_2**: Kriging \( (Y \sim \mathcal{P}G(m, k)) \) trained on \((X_2, y_2)\).
- **seqGPR**: Kriging \( ((Y_2 \sim \mathcal{P}G(m, k_1 + k_2)) \) trained on \((X_1, y_1)\) and \((X_2, y_2)\).

\[
\begin{align*}
\begin{cases}
Y_2(x_1, x_2, x_3, x_4) & = Y_1(x_1, x_2) + Z_2(x_1, x_2, x_3, x_4) \\
Z_2(x_1, x_2, \frac{x_1 + x_2}{2}, 0.2x_1 + 0.7) & = 0,
\end{cases}
\end{align*}
\]

- **Y_1 \sim \mathcal{P}G(m, k_1)**
- **Z_2 \sim \mathcal{P}G(0, k_2)** a process Red or P built from \( \tilde{Z}_2 \).
- **\tilde{Z}_2 \sim \mathcal{P}G(0, \kappa)** independent of \( Y_1 \) with robust parametrization \( \kappa : (\alpha, \alpha, \theta_3, \theta_4) \).

\( k, k_1, \) et \( \kappa \) are Matern \( \frac{5}{2} \) tensor-product.
Results

RMSE on a Sobol sequence of size 10000. Median and interquartile range of RMSE on 100 samples \((X_1, y_1)\) of size 20, \((X_2, y_2)\) of size 40. Red is better.

<table>
<thead>
<tr>
<th></th>
<th>K_2</th>
<th>K_tot</th>
<th>P</th>
<th>Red</th>
</tr>
</thead>
<tbody>
<tr>
<td>Médian</td>
<td>0.44</td>
<td>0.17</td>
<td>0.12</td>
<td>0.07</td>
</tr>
<tr>
<td>Interquartile range</td>
<td>0.05</td>
<td>0.09</td>
<td>0.05</td>
<td>0.02</td>
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Others studies show that the size of the training sample has no influence.
Results

RMSE on a Sobol sequence of size 10000. Median and interquartile range of RMSE on 100 samples \((X_1, y_1)\) of size 20, \((X_2, y_2)\) of size 40. \textbf{Red} is better.

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Others studies show that the size of the training sample has non influence.
Fan System in a cooling system

Input
Fan System in a cooling system

**Input**
Flowrate
Fan System in a cooling system

**Input**

Flowrate
Stagger_1
Stagger_2
Stagger_3
Stagger_4
Stagger_5
LChord_1
LChord_2
LChord_3
LChord_4
LChord_5
Fan System in a cooling system

**Input**
- Flowrate
- Stagger_1
- Stagger_2
- Stagger_3
- Stagger_4
- Stagger_5
- LChord_1
- LChord_2
- LChord_3
- LChord_4
- LChord_5
- Sweep_2
- Sweep_3
- Sweep_4
- Sweep_5
Fan System in a cooling system

**Input**
- Flowrate
- Stagger_1
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- Stagger_3
- Stagger_4
- Stagger_5
- LChord_1
- LChord_2
- LChord_3
- LChord_4
- LChord_5
- Sweep_2
- Sweep_3
- Sweep_4
- Sweep_5

**Output**: $\Delta P$
Data
# Data

**Step 1**

- Flowrate
- Stagger _1
- Stagger _2
- Stagger _3
- Stagger _4
- Stagger _5
- LChord _1
- LChord _2
- LChord _3
- LChord _4
- LChord _5
- Sweep _2=1
- Sweep _3=1
- Sweep _4=0.82
- Sweep _5=0.517645

<table>
<thead>
<tr>
<th></th>
<th>K_tot</th>
<th>Median</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total</td>
<td>44.6</td>
<td>5.9</td>
<td></td>
</tr>
<tr>
<td>Pressure</td>
<td>41.2</td>
<td>4.7</td>
<td></td>
</tr>
<tr>
<td>Red</td>
<td>42.1</td>
<td>4.6</td>
<td>27/32</td>
</tr>
</tbody>
</table>
# Data

## Step 1
- **Flowrate**
- **Stagger_1**
- **Stagger_2**
- **Stagger_3**
- **Stagger_4**
- **Stagger_5**
- **LChord_1**
- **LChord_2**
- **LChord_3**
- **LChord_4**
- **LChord_5**
- **Sweep_2=1**
- **Sweep_3=1**
- **Sweep_4=0.82**
- **Sweep_5=0.517645**

## Step 2
- **Flowrate**
- **Stagger_1**
- **Stagger_2**
- **Stagger_3**
- **Stagger_4**
- **Stagger_5**
- **LChord_1**
- **LChord_2**
- **LChord_3**
- **LChord_4**
- **LChord_5**
- **Sweep_2**
- **Sweep_3**
- **Sweep_4**
- **Sweep_5**

Median Range

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</tr>
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27/32
Data

**Step 1**
Flowrate
Stagger_1
Stagger_2
Stagger_3
Stagger_4
Stagger_5
LChord_1
LChord_2
LChord_3
LChord_4
LChord_5
Sweep_2=1
Sweep_3=1
Sweep_4=0.82
Sweep_5=0.517645

**Step 2**
Flowrate
Stagger_1
Stagger_2
Stagger_3
Stagger_4
Stagger_5
LChord_1
LChord_2
LChord_3
LChord_4
LChord_5
Sweep_2
Sweep_3
Sweep_4
Sweep_5

- 30 models
- \( X_1 \): 50 points in \([0, 1]\)
- \( X_2 \): 50 points in \([0, 1]\)
Data

**Step 1**
- Flowrate
- Stagger_1
- Stagger_2
- Stagger_3
- Stagger_4
- Stagger_5
- LChord_1
- LChord_2
- LChord_3
- LChord_4
- LChord_5
- Sweep_2\(=1\)
- Sweep_3\(=1\)
- Sweep_4\(=0.82\)
- Sweep_5\(=0.517645\)

**Step 2**
- Flowrate
- Stagger_1
- Stagger_2
- Stagger_3
- Stagger_4
- Stagger_5
- LChord_1
- LChord_2
- LChord_3
- LChord_4
- LChord_5
- Sweep_2
- Sweep_3
- Sweep_4
- Sweep_5

- 30 models
- \(X_1\) : 50 points in \([0, 1]\)^{11}
- \(X_2\) : 50 points in \([0, 1]\)^{15}

<table>
<thead>
<tr>
<th></th>
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<tbody>
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<td>(K_{tot})</td>
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<td>(Red)</td>
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Table des matières

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- Kriking
- New Model
- Correction process

- Parameters estimation

- Applications
  - Analytic example in dimension 4
  - Industrial Test Cases

- Conclusion
Conclusion

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- Nullity constraints: 2 candidates (Red and P)
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- EM Algorithm to estimate the model EM pour estimer les paramètres
Conclusion

- Autoregressive model inspired by multi-fidelity models
- Nullity constraints: 2 candidates (Red and P)
- EM Algorithm to estimate the model EM pour estimer les paramètres
- Good results on the test case
Thank you for your attention
References I


References II